Linear Independence

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A set of vectors is linearly independent if the only solution to $c_1V_1 + c_2V_2 \dots + c_kV_k = 0$ where $c_1, c_2 \dots c_n$ is any scalar and, $v_1, v_2 \dots v_n$ are vectors, is $c_i = 0$ for all i.

Given a set of vectors, we can determine if they are linearly independent by writing the vectors as the columns of the matrix A, and solving $AX = 0$. If there are any non-zero solutions, then the vectors are linearly dependent. If the only solution is $X = 0$, then they are linearly independent.

Rank of a Matrix

Sub matrices of a matrix:

Let A be a $m \times n$ matrix, then a matrix obtained by deleting some rows and columns from A is called a submatrix of A.

The Rank of a matrix is the order of the largest square matrix whose determinant is not zero.

Thus a number 'r' is said to be the rank of a matrix A if it possess the following two properties.

i). There is atleast one square submatrix A of order 'r' whose determinant is not equal to zero.

ii) If the matrix A contains any square submatrix of order $r+1$ then the determinent of every square submatrix of order $r+1$ should be zero.

* Find the rank of the following matrices.

a) $A = \begin{bmatrix} 2 & 4 & 1 \\ 3 & 2 & 4 \\ 3 & 1 & 5 \end{bmatrix}$

Sol: $|A| = 2[10-4]-4[15-12]+1[3-6]$

$= 2\times6 - 4\times3 + 1\times-3$

$= 12 - 12 - 3$

$= -3 \neq 0$

Hence, rank $= 3$

b) $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{bmatrix}$

Sol: $|A| = 1(21-20) - 2(14-12) + 3(10-9)$

$= 1(1) - 2(2) + 3(1)$

$= 1 - 4 + 3 = \underline{\underline{0}}$

Now, consider the square submatrix $\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$.

$\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = 3 - 4 = -1 \neq 0$

$\therefore$ Rank $= \underline{\underline{2}}$

c) $\begin{bmatrix} -2 & 1 & 3 & 4 \\ 0 & 1 & 1 & 2 \\ 1 & 3 & 4 & 7 \end{bmatrix}$

Sol: Consider the square submatrix $\begin{bmatrix} -2 & 1 & 3 \\ 0 & 1 & 1 \\ 1 & 3 & 4 \end{bmatrix}$

$\begin{vmatrix} -2 & 1 & 3 \\ 0 & 1 & 1 \\ 1 & 3 & 4 \end{vmatrix} = -2[4-3] - 1[0-1] + 3[0-1]$

$= -2 + 1 - 3 = -4 \neq 0$

$\therefore$ Rank $= \underline{\underline{3}}$